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On Euclidean invariance of algebraic Reynolds stress models in turbulence

By J. WEIS AND K. HUTTER

Darmstadt University of Technology, Department of Mechanics, Hochschulstr. 1, 64289 Darmstadt, Germany

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This article shows how Euclidean invariance can be preserved in the so-called algebraic Reynolds stress model (ARSM) approximation. This approximation is used to reduce the transport equation for the Reynolds stresses to an explicit algebraic relation. A number of known models, which make use of this approximation, are not form-invariant under transformations to rotating coordinate systems. A simple extension is presented to show how this artifact can be removed.

1. Introduction

That the physical laws must be the same for observers in different coordinate systems is an almost trivial statement. The choice of the coordinate system cannot influence the physics, because it is only a mathematical tool to describe the physics. At a more sophisticated level this simple statement is expressed in terms of invariance requirements of the laws of mechanics under a change of observer, known as an Euclidean transformation. In short, the balance laws of mass, momentum and energy are form-invariant under Euclidean transformations. However, they can depend on the motion of the system, whenever the frame of reference is non-inertial.

Gatski & Speziale (1993), Girimaji (1996) and Wallin & Johansson (2000) employ the so-called ARSM approximation, which is used to derive an algebraic relation from the transport equation for the Reynolds stress tensor. However, their approximations do not lead to Euclidean-invariant models. Wallin & Johansson (2000) write, "the adequacy of the ARSM approach is coupled to the choice of a coordinate system . . .". Fundamentally this cannot be true: there is neither a better nor a worse coordinate system, though sometimes it might be more convenient to make use of one in favour of another.

The reason for this artifact is an error in the derivation of the ARSM approximation. A non-objective term is neglected which results in a non-invariant model. Girimaji (1997) corrects this. Unfortunately he makes use of a non-objective expression in the formulation of the pressure/rate-of-strain tensor: the last term of equation (27) is modelled with the non-objective antisymmetric part of the velocity gradient. There is also a recent paper by Gatski & Jongen (2000) which attempts exactly what we present in this paper, but due to an error in the calculation or a misprint their final equation for the anisotropy tensor is not Euclidean objective: equation (123) in which the last term of the first line contains a vorticity measure that is not form-invariant. This paper shows how this can be corrected, and so can be regarded as a commentary on the original papers of Girimaji (1997) and Gatski & Jongen (2000).

2. Transformation of variables

A crucial element in demonstrating the necessary invariance properties of ARSMs is to show how the various field quantities arising in the Reynolds stress balance transform under change of reference frame. For simplicity, only transformations between rotating systems with the same origin and with an arbitrary time-dependent angular velocity ω_k are considered. The space coordinates x_l then transform as follows:

$$x_i^* = O_{il} x_l, \quad \text{with} \quad O_{il}^T = O_{il}^{-1} \quad \text{and} \quad \Omega_{ij} := O_{il}^T \dot{O}_{lj} = \epsilon_{ilj}\omega_l. \tag{2.1}$$

O_{il} is an orthogonal time-dependent transformation and Ω_{ij}, usually called the rotation tensor, is skew symmetric, $\Omega_{ij} = -\Omega_{ji}$. All variables in the transformed system are identified with an asterisk. The most relevant physical variables for algebraic Reynolds stress modelling are

$$\left.\begin{aligned} v_i &: \text{velocity vector},\\ D_{ij} = \tfrac{1}{2}(\partial_j v_i + \partial_i v_j) &: \text{symmetric part of velocity gradient},\\ W_{ij} = \tfrac{1}{2}(\partial_j v_i - \partial_i v_j) &: \text{antisymmetric part of velocity gradient},\\ k &: \text{turbulent kinetic energy},\\ \epsilon &: \text{dissipation of turbulent kinetic energy},\\ R_{ij} &: \text{Reynolds stress tensor},\\ a_{ij} = -\tfrac{1}{3}\,\delta_{ij} - R_{ij}/(2k) &: \text{anisotropy tensor}. \end{aligned}\right\} \tag{2.2}$$

We will restrict attention to a viscous incompressible fluid with constant viscosity μ and constant density ρ. Transformation according to equation (2.1) implies the following variable connections between the new and old systems:

$$\left.\begin{aligned} k^* = k, \quad \epsilon^* = \epsilon,\\ D_{ij}^* = O_{il} D_{lm} O_{mj}^T, \quad R_{ij}^* = O_{il} R_{lm} O_{mj}^T, \quad a_{ij}^* = O_{il} a_{lm} O_{mj}^T. \end{aligned}\right\} \tag{2.3}$$

Moreover, gradients transform as

$$\partial_l^* = O_{lm}\partial_m. \tag{2.4}$$

For the antisymmetric part of the velocity gradient one may deduce

$$W_{ij}^* = O_{il}(W_{lm} + \Omega_{lm}) O_{mj}^T, \tag{2.5}$$

and for the total time derivatives of R_{ij} and a_{ij} one obtains

$$\dot{R}_{ij}^* = O_{il}(\dot{R}_{lm} + \Omega_{lk} R_{km} - R_{lk}\Omega_{km}) O_{mj}^T, \tag{2.6}$$

$$\dot{a}_{ij}^* = O_{il}(\dot{a}_{lm} + \Omega_{lk} a_{km} - a_{lk}\Omega_{km}) O_{mj}^T. \tag{2.7}$$

Note that the right-hand sides of equations (2.5), (2.6) and (2.7) involve system-dependent terms whilst (2.3) does not.

3. Objective variables and invariance of transport equations

A scalar s, a vector b_i and a tensor c_{ij}, that transform under Euclidean transformations according to

$$s^* = s, \tag{3.1}$$

$$b_i^* = O_{im} b_m, \tag{3.2}$$

$$c_{ij}^* = O_{il} c_{lm} O_{mj}^T \tag{3.3}$$

are called objective variables, because they transform like geometric objects. Some physical variables are objective (compare equation (2.3)), but some are not, such as the velocity vector. It is not a 'geometrical' vector, and it cannot be interpreted as an arrow, which can simply be 'rotated' from one system into another. Its definition depends on the system: the total time derivative of the space vector from a particle in a certain system.

Every physical law, a transport equation or balance law, must have the same form in different systems, i.e. it must be form-invariant, though it may happen that new terms enter the equation, which depend on the motion of the particular frame. For example the Coriolis force is such a frame-dependent term in the momentum equation. It is obvious that a transport equation constructed with objective terms results in an invariant equation. Therefore, when postulating a closure relation for an objective quantity, it is mandatory to express it in objective variables without using the non-objective quantities.

In the following it will be assumed that the transformed system, denoted by the asterisk is inertial. Space coordinates then transform into different rotating systems according to

$$x_i = O^T_{il} x^*_l. \tag{3.4}$$

The angular velocity ω^*_i and the rotation tensor Ω^*_{ij} of the inertial system are by definition zero. Any other transformed system can now be identified with a certain angular velocity ω_i and an associated rotation tensor $\Omega_{ij} = O^T_{il}\dot{O}_{lj} = \epsilon_{ilj}\omega_l$. Under such circumstances the variables

$$\tilde{W}_{ij} := W_{ij} + \Omega_{ij}, \tag{3.5}$$

$$\tilde{\dot{R}}_{ij} := \dot{R}_{ij} + \Omega_{il}R_{lj} - R_{il}\Omega_{lj}, \tag{3.6}$$

$$\tilde{\dot{a}}_{ij} := \dot{a}_{ij} + \Omega_{il}a_{lj} - a_{il}\Omega_{lj} \tag{3.7}$$

are objective, as follows immediately from equations (2.5), (2.6) and (2.7). The symmetric part of the velocity gradient D_{ij}, the Reynolds stress tensor R_{ij}, the anisotropy tensor a_{ij}, the turbulent kinetic energy k and the dissipation rate of the turbulent kinetic energy ϵ are all objective variables.

Now, the transport equation for the Reynolds stress tensor R_{ij}, which is derived after Reynolds averaging, can be transformed from the inertial system into any other frame according to the transformation rule (3.4):

$$\rho\tilde{\dot{R}}_{ij} = -R_{il}(D_{lj} - \tilde{W}_{lj}) - (D_{il} + \tilde{W}_{il})R_{lj} + \left(\partial_l \Phi^{R_{ij}}_l - \epsilon_{ij} - \Pi_{ij}\right). \tag{3.8}$$

The flux of the Reynolds stresses is denoted by $\Phi^{R_{ij}}_l$, the sources by Π_{ij} and the dissipation by ϵ_{ij}. The tensor Π_{ij} is also known as the pressure/rate-of-strain tensor. In equation (3.8) half of the system-dependent term (resulting from the Coriolis force) is included in the objective time derivative $\tilde{\dot{R}}$ and the other half in the objective antisymmetric part of the velocity gradient $\tilde{W}_{ij}$. Because the whole expression (3.8) is objective and the term on the left-hand side and the first two terms on the right-hand side are also objective, one can conclude that $(\partial_l \Phi^{R_{ij}}_l - \epsilon_{ij} - \Pi_{ij})$ is also an objective term, which has to be modelled by using objective variables. This was observed by Lumley (1970).

4. System dependence of constitutive relations

It is usually assumed that material laws do not depend on the rotation of the system, ω_i. This means that in every system the material should show the same behaviour. This is quite a good assumption as long as the relaxation time of the material is large compared with the typical time scale of the flow.

In turbulence modelling one is dealing with closures and not with material laws. In the broadest sense, subgrid flow properties are parameterized, so turbulence itself is not a material property in the usual sense. In such flows the characteristic turbulent time scale can be comparable with the typical time scale of the flow, implying that the rotation of the system can influence the turbulent closure functionals. This means that objective quantities, which depend on the rotation of the reference frame, may enter such functional relations. It implies for example that the Reynolds stress tensor R_{ij} can depend on the 'effective' rotation tensor $\tilde{W}_{ij}$, see equation (3.5), which is an objective variable and depends on the rotation of the system. Nevertheless, it must be ascertained that the whole set of field equations remains invariant under Euclidean transformations, because this is equivalent to different observers looking at the same flow.

5. Closures and the ARSM approximation

With the aid of the ARSM approximation first presented by Pope (1975) a differential Reynolds stress equation can be reduced to an algebraic relationship. Pope considered only two-dimensional mean flow and no rotation of the system. His method was extended by Gatski & Speziale (1993) to three-dimensional mean flow in rotating systems. Girimaji (1996) published an improved version for two-dimensional mean flow and rotating systems.

In these approaches the pressure–strain correlation Π_{ij} is modelled as a tensorial relation, which is quasi-linear in the anisotropy tensor a_{ij}. Gatski & Speziale (1993) and Girimaji (1996), for example, use the representation

$$\frac{1}{\rho k}\Pi_{ij} \approx -\left(c_{10}\frac{\epsilon}{k} - 2c_{11}a_{nl}D_{ln}\right)a_{ij} + c_2 D_{ij} + c_3\left(D_{il}a_{lj} + a_{il}D_{lj} - \tfrac{2}{3}a_{nl}D_{ln}\delta_{ij}\right) + c_4(\tilde{W}_{il}a_{lj} - a_{il}\tilde{W}_{lj}), \quad (5.1)$$

which is objective since it is modelled with objective variables D_{ij}, $\tilde{W}_{ij}$, k and ϵ; c_{10}, c_{11}, c_2, c_3 and c_4 are constants. The dissipation-rate tensor ϵ_{ij} is assumed to be isotropic and also objective:

$$\epsilon_{ij} = -\tfrac{2}{3}\rho\epsilon\delta_{ij}. \quad (5.2)$$

To arrive at an implicit algebraic expression for the Reynolds stress tensor the total time derivative of the anisotropy tensor $\dot{a}_{ij}$ and the diffusion term of the Reynolds stesses are neglected. That is,

$$\dot{a}_{ij} \approx 0, \quad (5.3)$$

$$\partial_l \Phi_l^{R_{ij}} + \tfrac{R_{ij}}{2\rho k}\partial_l \Phi_l^{R_{kk}} \approx 0. \quad (5.4)$$

This is called the ARSM approximation. However, this is not invariant modelling, because the total time derivative in the approximation (5.3) is not objective. Each observer has their own approximation, which should not be the case. This implies that the so-called 'effective' mean rotation tensor is not objective and therefore the whole model is not invariant. The same remark applies to another proposition by Wallin & Johansson (2000). An appropriate modelling would be

$$\tilde{\dot{a}}_{ij} = \dot{a}_{ij} + \Omega_{il} a_{lj} - a_{il} \Omega_{lj} \approx 0. \tag{5.5}$$

Together with equation (5.4) this is an objective approximation and therefore ensures invariant modelling. If the closures (5.1) and (5.2) and the approximations (5.4) and (5.5) are inserted in equation (3.8) and the emerging equation is rearranged, an implicit algebraic equation for the anisotropy tensor a_{ij} is obtained as follows:

$$[2B_1 a_{ln} \hat{D}_{nl} - 1] a_{ij} = \tfrac{1}{3} B_2 \hat{D}_{ij} + B_3 \left[\hat{D}_{il} a_{lj} + a_{il} \hat{D}_{lj} - \tfrac{2}{3} \hat{D}_{nl} a_{ln} \delta_{ij} \right] + B_4 [\hat{W}_{il} a_{lj} - a_{il} \hat{W}_{lj}], \tag{5.6}$$

with newly defined constants

$$B_1 = \frac{2 + c_{11}}{c_{10} - 2}, \quad B_2 = \frac{4 - 3c_2}{c_{10} - 2}, \quad B_3 = \frac{2 - c_3}{c_{10} - 2}, \quad B_4 = \frac{2 - c_4}{c_{10} - 2}, \tag{5.7}$$

and the dimensionless objective velocity gradients

$$\hat{D}_{ij} = \frac{k}{\epsilon} D_{ij} \quad \text{and} \quad \hat{W}_{ij} = \frac{k}{\epsilon} \tilde{W}_{ij} = \frac{k}{\epsilon} (W_{ij} + \Omega_{ij}). \tag{5.8}$$

Now the implicit algebraic equation (5.6) for the anisotropy tensor a_{ij} can be solved as suggested by Gatski & Speziale (1993), Girimaji (1996) or Wallin & Johansson (2000) yielding an explicit expression for the anisotropy tensor a_{ij} and also for the Reynolds stress tensor R_{ij}. For two-dimensional mean flow this can be done exactly. For general three-dimensional mean flows equation (5.6) cannot be solved analytically: a complicated nonlinear system of equations for the five unkown independent terms of the anisotropy tensor a_{ij} is obtained, which is not amenable to exact solutions. Again approximations must be introduced. However, up to now it is not clear which approximation is the best.

Comparing these results with those of Gatski & Speziale (1993), Girimaji (1996) or Wallin & Johansson (2000), it can be seen that to make the proposed ARSM closures form-invariant, one need only change the expressions involving the vorticity tensor: instead of $(k/\epsilon)\, W_{ij}$ the dimensionless effective mean rotation tensor $\hat{W}_{ij}$, defined in equation (5.8), must be used. Nothing else changes.

Girimaji (1997) and Gatski & Jongen (2000) attempt to do exactly what we present in this paper. Although Girimaji (1997) has a somewhat different notation, both papers make use of an objective ARSM approximation. Unfortunately they do not arrive at a form-invariant model. Their final equations contain vorticity terms that are not objective. While Gatski & Jongen (2000) have an error in their calculation, Girimaji (1997) models the pressure/rate-of-strain tensor with the non-objective antisymmetric part of the velocity gradient.

6. Conclusions

It has been shown how the algebraic Reynolds stress models of Gatski & Speziale (1993), Girimaji (1996) and Wallin & Johansson (2000) can be changed into form-invariant models. If the proposed modifications are implemented, the choice of the coordinate system can no longer affect the adequacy of the models. The correction has no effect for non-rotating systems, but it involves a new calibration whenever rotating systems are considered.

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